Universität Konstanz
Fachbereich Mathematik und Statistik
Prof. Dr. Salma Kuhlmann
Simon Müller
Michele Serra
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Exercises for the course "Linear Algebra I"

Sheet 13

Hand in your solutions on Thursday, 6. Februar 2020, 09:55, in the Postbox of your Tutor in F4. Please try to write your solutions clear and readable, write your name and the name of your tutor on each sheet, and staple the sheets together.

Exercise 13.1 ***Beweismechanikaufgabe*** (*4 points*)

Bitte gehen Sie in dieser Aufgabe nach den Regeln der Beweismechanik vor und geben Ihre Lösung auf einem separaten Blatt in den Briefkasten mit der Aufschrift „Beweismechanikaufgaben" ab. Ihnen unbekannte Begriffe und Symbole können Sie in der Beweismechanik nachschlagen.

Sei K ein Körper, V ein endlichdimensionaler K-Vektorraum und $U, W \subset V$ Unterräume. Zeigen Sie

(i) $(U + W)^\circ = U^\circ \cap W^\circ$
(ii) $U^\circ + W^\circ \subset (U \cap W)^\circ$.
Hinweis: Es gilt auch $(U \cap W)^\circ \subset U^\circ + W^\circ$. *Dies müssen Sie jedoch hier nicht beweisen.*

Exercise 13.2 (*4 points*)

Let $\mathcal{E} = \{e_1, ..., e_5\}$ be the standard basis of $\mathbb{R}^{5,}$ and let W be the subspace of $\mathbb{R}^5$ that is generated by

$$\alpha_1 = e_1 + 2e_2 + e_3$$
$$\alpha_2 = e_2 + 3e_3 + 3e_4 + e_5$$
$$\alpha_3 = e_1 + 4e_2 + 6e_3 + 4e_4 + e_5$$

Find a basis for W^0.

Definition

In Linear Algebra II we will consider the so-called *determinant*. It is a mapping that assigns to any $A \in \mathrm{Mat}_{n\times n}(K)$ an element $\det(A) \in K$. It can for instance be shown that a matrix A is invertible if and only if $\det(A) \neq 0$.

We give explicit formulas for the determinant in case $n = 2$ and $n = 3$. Use them to solve Excercise 13.3 and 13.4 below.

- If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(K)$, then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.
- If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{Mat}_{3\times 3}(M)$, then

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Exercise 13.3 (*4 points*)

Let K be a field and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(K)$.

(a) Show that A is invertible if and only if $\det(A) \neq 0$.
(b) Let A be invertible. Give an explicit formula for the inverse of A.

Exercise 13.4 *(4 points)*

Let K be a field and $A \in \mathrm{Mat}_{3\times 3}(K)$. Let $B \in \mathrm{Mat}_{3\times 3}(K)$ be a matrix that is obtained from A by a single row operation of type 1, and correspondingly $C, D \in \mathrm{Mat}_{3\times 3}(K)$ matrices that are obtained from A by a single type 2, respectively single type 3 operation. Show that:

(a) $\det B = -\det(A)$

(b) $\det C = \lambda \det(A)$, where λ is the scalar with which we multiplied a row of A.

(c) $\det D = \det(A)$

Hint: if you consider the i-th row of A, the other two rows can be indexed as $i+1$ and $i+2$ with the following convention:

- for $i = 1$ we have $i+1 = 2$ and $i+2 = 3$ (as usual);
- for $i = 2$ we have $i+1 = 3$ and $i+2 = 1$;
- for $i = 3$ we have $i+1 = 1$ and $i+2 = 2$.